# Geometry. IMO

Данный листок содержит все задачи по геометрии, которые предлагались на Международной математической олимпиаде (IMO) начиная с 2005 года.

Международная математическая олимпиада проходит в два дня. Задачи 1, 2, 3 даются в первый день, задачи 4, 5, 6 — во второй. В варианте каждого дня задачи обычно расположены по возрастанию сложности; таким образом, задачи 1 и 4 являются «простыми», задачи 2 и 5 — «средней сложности», задачи 3 и 6 — самые трудные.

Принцип нумерации задач листка: задача **15.3** предлагалась в 2015 году под номером 3.

## Problems 1 and 4

**15.4.** Triangle $ABC$ has circumcircle $\Omega$ and circumcenter $O$. A circle $\Gamma$ with center $A$ intersects the segment $BC$ at points $D$ and $E$, such that $B$, $D$, $E$, and $C$ are all different and lie on line $BC$ in this order. Let $F$ and $G$ be the points of intersection of $\Gamma$ and $\Omega$, such that $A$, $F$, $B$, $C$, and $G$ lie on $\Omega$ in this order. Let $K$ be the second point of intersection of the circumcircle of triangle $BDF$ and the segment $AB$. Let $L$ be the second point of intersection of the circumcircle of triangle $CGE$ and the segment $CA$.

Suppose that the lines $FK$ and $GL$ are different and intersect at the point $X$. Prove that $X$ lies on the line $AO$.

**14.4.** Points $P$ and $Q$ lie on side $BC$ of acute-angled triangle $ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points $M$ and $N$ lie on lines $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$, and $Q$ is the midpoint of $AN$. Prove that lines $BM$ and $CN$ intersect on the circumcircle of triangle $ABC$.

**13.4.** Let $ABC$ be an acute-angled triangle with orthocenter $H$, and let $W$ be a point on the side $BC$, lying strictly between $B$ and $C$. The points $M$ and $N$ are the feet of the altitudes from $B$ and $C$, respectively. Denote by $\omega_1$ the circumcircle of $BWN$, and let $X$ be the point on $\omega_1$ such that $WX$ is a diameter of $\omega_1$. Analogously, denote by $\omega_2$ the circumcircle of $CWM$, and let $Y$ be the point on $\omega_2$ such that $WY$ is a diameter of $\omega_2$. Prove that $X, Y$ and $H$ are collinear.

**12.1.** Given triangle $ABC$ the point $J$ is the centre of the excircle opposite the vertex $A$. This excircle is tangent to the side $BC$ at $M$, and to the lines $AB$ and $AC$ at $K$ and $L$, respectively. The lines $LM$ and $BJ$ meet at $F$, and the lines $KM$ and $CJ$ meet at $G$. Let $S$ be the point of intersection of the lines $AF$ and $BC$, and let $T$ be the point of intersection of the lines $AG$ and $BC$.

Prove that $M$ is the midpoint of $ST$.

**10.4.** Let $P$ be a point inside the triangle $ABC$. The lines $AP$, $BP$ and $CP$ intersect the circumcircle $\Gamma$ of triangle $ABC$ again at the points $K$, $L$ and $M$ respectively. The tangent to $\Gamma$ at $C$ intersects the line $AB$ at $S$. Suppose that $SC = SP$. Prove that $MK = ML$.

**09.4.** Let $ABC$ be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides $BC$ and $CA$ at $D$ and $E$, respectively. Let $K$ be the incentre of triangle $ADC$. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

**08.1.** An acute-angled triangle $ABC$ has orthocentre $H$. The circle passing through $H$ with centre the midpoint of $BC$ intersects the line $BC$ at $A_1$ and $A_2$. Similarly, the circle passing through $H$ with centre the midpoint of $CA$ intersects the line $CA$ at $B_1$ and $B_2$, and the circle passing through $H$ with centre the midpoint of $AB$ intersects the line $AB$ at $C_1$ and $C_2$. Show that $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ lie on a circle.

**07.4.** In triangle $ABC$ the bisector of angle $BCA$ intersects the circumcircle again at $R$, the perpendicular bisector of $BC$ at $P$, and the perpendicular bisector of $AC$ at $Q$. The midpoint of $BC$ is $K$ and the midpoint of $AC$ is $L$. Prove that the triangles $RPK$ and $RQL$ have the same area.

**06.1.** Let $ABC$ be a triangle with incenter $I$. A point $P$ in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geqslant AI$, and that equality holds if and only if $P = I$.

**05.1.** Six points are chosen on the sides of an equilateral triangle $ABC$: $A_1$, $A_2$ on $BC$, $B_1$, $B_2$ on $CA$ and $C_1$, $C_2$ on $AB$, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines $A_1B_2$, $B_1C_2$ and $C_1A_2$ are concurrent.

## Problems 2 and 5

**12.5.** Let $ABC$ be a triangle with $\angle BCA = 90°$, and let $D$ be the foot of the altitude from $C$. Let $X$ be a point in the interior of the segment $CD$. Let $K$ be the point on the segment $AX$ such that $BK = BC$. Similarly, let $L$ be the point on the segment $BX$ such that $AL = AC$. Let $M$ be the point of intersection of $AL$ and $BK$.

Show that $MK = ML$.

**11.2.** Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line $\ell$ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* $P$ until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, $Q$, takes over as the new pivot, and the line now rotates clockwise about $Q$, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point $P$ in $\mathcal{S}$ and a line $\ell$ going through $P$ such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

**10.2.** Let $I$ be the incentre of triangle $ABC$ and $\Gamma$ be its circumcircle. Let the line $AI$ intersects $\Gamma$ again at $D$. Let $E$ be a point on the arc $BDC$, and $F$ a point on the side $BC$ such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

Finally, let $G$ be the midpoint of segment $IF$. Prove that the lines $DG$ and $EI$ intersect on $\Gamma$.

**09.2.** Let $ABC$ be a triangle with circumcentre $O$. The points $P$ and $Q$ are interior points of the sides $CA$ and $AB$, respectively. Let $K, L$ and $M$ be the midpoints of the segments $BP, CQ$ and $PQ$, respectively, and let $\Gamma$ be the circle passing through $K, L$ and $M$. Suppose that the line $PQ$ is tangent to the circle $\Gamma$. Prove that $OP = OQ$.

**07.2.** Consider five points $A$, $B$, $C$, $D$ and $E$ such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let $\ell$ be a line passing through $A$. Suppose that $\ell$ intersects the interior of the segment $DC$ at $F$ and intersects line $BC$ at $G$. Suppose also that $EF = EG = EC$. Prove that $\ell$ is the bisector of $\angle DAB$.

**05.5.** Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $BC$ not parallel with $DA$. Let two variable points $E$ and $F$ lie of the sides $BC$ and $DA$, respectively and satisfy $BE = DF$. The lines $AC$ and $BD$ meet at $P$, the lines $BD$ and $EF$ meet at $Q$, the lines $EF$ and $AC$ meet at $R$.

Prove that the circumcircles of the triangles $PQR$, as $E$ and $F$ vary, have a common point other than $P$.

## Problems 3 and 6

**15.3.** Let $ABC$ be an acute triangle with $AB > AC$. Let $\Gamma$ be its circumcircle, $H$ its orthocenter, and $F$ the foot of the altitude from $A$. Let $M$ be the midpoint of $BC$. Let $Q$ be the point on $\Gamma$ such that $\angle HQA = 90°$, and $K$ be the point on $\Gamma$ such that $\angle HKQ = 90°$. Assume that the points $A$, $B$, $C$, $K$, and $Q$ are all different, and lie on $\Gamma$ in this order.

Prove that the circumcircles of triangles $KQH$ and $FKM$ are tangent to each other.

**14.3.** Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point $H$ is the foot of the perpendicular from $A$ to $BD$. Points $S$ and $T$ lie on sides $AB$ and $AD$, respectively, such that $H$ lies inside triangle $SCT$ and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line $BD$ is tangent to the circumcircle of triangle $TSH$.

**13.3.** Let the excircle of triangle $ABC$ opposite the vertex $A$ be tangent to the side $BC$ at the point $A_1$. Define the points $B_1$ on $CA$ and $C_1$ on $AB$ analogously, using the excircles opposite $B$ and $C$, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle $ABC$. Prove that triangle $ABC$ is right-angled.

**11.6.** Let $ABC$ be an acute triangle with circumcircle $\Gamma$. Let $\ell$ be a tangent line to $\Gamma$, and let $\ell_a, \ell_b$ and $\ell_c$ be the lines obtained by reflecting $\ell$ in the lines $BC$, $CA$ and $AB$, respectively. Show that the circumcircle of the triangle determined by the lines $\ell_a, \ell_b$ and $\ell_c$ is tangent to the circle $\Gamma$.

**08.6.** Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles $ABC$ and $ADC$ by $\omega_1$ and $\omega_2$ respectively. Suppose that there exists a circle $\omega$ tangent to the ray $BA$ beyond $A$ and to the ray $BC$ beyond $C$, which is also tangent to the lines $AD$ and $CD$. Prove that the common external tangents of $\omega_1$ and $\omega_2$ intersect on $\omega$.

**06.6.** Assign to each side $b$ of a convex polygon $P$ the maximum area of a triangle that has $b$ as a side and is contained in $P$. Show that the sum of the areas assigned to the sides of $P$ is at least twice the area of $P$.