Примеры вычисления интегралов, содержащих тригонометрические функции $\operatorname{tg} x$ и $\operatorname{ctg} x$.

$$\int \operatorname{tg} x\,dx \qquad \int \operatorname{tg}^2 x\,dx \qquad \int \operatorname{tg}^3 x\,dx \qquad \int \operatorname{tg}^4 x\,dx \qquad \int \operatorname{tg}^5 x\,dx \qquad \int \operatorname{tg}^6 x\,dx$$

$$\int \operatorname{ctg} x\,dx \qquad \int \operatorname{ctg}^2 x\,dx \qquad \int \operatorname{ctg}^3 x\,dx \qquad \int \operatorname{ctg}^4 x\,dx \qquad \int \operatorname{ctg}^5 x\,dx \qquad \int \operatorname{ctg}^6 x\,dx \qquad \int \frac{dx}{\operatorname{ctg}^6 x}$$

$$\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x| + C$$

$$\int \operatorname{tg}^2 x\,dx = \int\left(\frac{1}{\cos^2 x} - 1\right)dx = \int \frac{dx}{\cos^2 x} - \int dx = \operatorname{tg} x - x + C$$

$$\int \operatorname{tg}^3 x\,dx = \int \frac{\sin^3 x}{\cos^3 x}dx = \int \frac{\sin^2 x}{\cos^3 x}\cdot \sin x\,dx = -\int \frac{\left(1-\cos^2 x\right)}{\cos^3 x}d(\cos x) = \int\left(\frac{1}{\cos x} - \frac{1}{\cos^3 x}\right)d(\cos x) =$$

$$= \ln|\cos x| + \frac{1}{2\cos^2 x} + C_1 = \ln|\cos x| + \frac{1}{2}\cdot\left(\operatorname{tg}^2 x + 1\right) + C_1 = \ln|\cos x| + \frac{\operatorname{tg}^2 x}{2} + C$$

$$\int \operatorname{tg}^4 x\,dx = \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t + \pi k,\quad k \in Z \\ dx = \dfrac{dt}{1+t^2} \end{array}\right] = \int \frac{t^4 dt}{1+t^2} = \int\left(-1 + t^2 + \frac{1}{1+t^2}\right)dt = -t + \frac{t^3}{3} + \operatorname{arctg} t + C_1 =$$

$$= -\operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + x - \pi k + C_1 = x - \operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + C$$

$$\int \operatorname{tg}^5 x\,dx = \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t + \pi k,\quad k \in Z \\ dx = \dfrac{dt}{1+t^2} \end{array}\right] = \int \frac{t^5 dt}{1+t^2} = \int\left(t^3 - t + \frac{t}{1+t^2}\right)dt =$$

$$= \int t^3 dt - \int t\,dt + \frac{1}{2}\cdot\int \frac{d\left(1+t^2\right)}{1+t^2} = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2}\ln\left|1+t^2\right| + C = \frac{\operatorname{tg}^4 x}{4} - \frac{\operatorname{tg}^2 x}{2} + \frac{1}{2}\ln\left|1+\operatorname{tg}^2 x\right| + C =$$

$$= \frac{\operatorname{tg}^2 x}{2}\cdot\left(\frac{\operatorname{tg}^2 x}{2} - 1\right) + \frac{1}{2}\ln\left|\frac{1}{\cos^2 x}\right| + C = \frac{\operatorname{tg}^2 x}{2}\cdot\left(\frac{\operatorname{tg}^2 x}{2} - 1\right) - \ln|\cos x| + C$$

$$\int \operatorname{tg}^6 x\,dx = \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t + \pi k,\quad k \in Z \\ dx = \dfrac{dt}{1+t^2} \end{array}\right] = \int \frac{t^6 dt}{1+t^2} = \int\left(t^4 - t^2 + 1 - \frac{1}{1+t^2}\right)dt =$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + t - \int d(\operatorname{arctg} t) = \frac{t^5}{5} - \frac{t^3}{3} + t - \operatorname{arctg} t + C_1 =$$

$$= \frac{\operatorname{tg}^5 x}{5} - \frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x - x + \pi k + C_1 = -x + \frac{\operatorname{tg}^5 x}{5} - \frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x + C$$

$$\int \operatorname{ctg} x\,dx=\int\frac{\cos x}{\sin x}dx=\int\frac{d(\sin x)}{\sin x}=\ln|\sin x|+C$$

$$\int \operatorname{ctg}^2 x\,dx=\int\left(\frac{1}{\sin^2 x}-1\right)dx=\int\frac{dx}{\sin^2 x}-\int dx=-\operatorname{ctg} x-x+C$$

$$\int \operatorname{ctg}^3 x\,dx=\int\frac{\cos^3 x}{\sin^3 x}dx=\int\frac{\cos^2 x}{\sin^3 x}\cdot\cos x\,dx=\int\frac{\left(1-\sin^2 x\right)}{\sin^3 x}d(\sin x)=\int\left(\frac{1}{\sin^3 x}-\frac{1}{\sin x}\right)d(\sin x)=$$
$$=-\frac{1}{2\sin^2 x}-\ln|\sin x|+C_1=-\frac{1}{2}\cdot\left(\operatorname{ctg}^2 x+1\right)-\ln|\sin x|+C_1=-\frac{\operatorname{ctg}^2 x}{2}-\ln|\sin x|+C$$

$$\int \operatorname{ctg}^4 x\,dx=\left[\begin{array}{l} t=\operatorname{ctg} x\\ x=\operatorname{arcctg} t+\pi k,\ \ k\in Z\\ dx=-\dfrac{dt}{1+t^2}\end{array}\right]=-\int\frac{t^4 dt}{1+t^2}dx=-\int\left(t^2-1+\frac{1}{1+t^2}\right)dx=$$
$$=-\left(\frac{t^3}{3}-t-\operatorname{arcctg} t+\pi k-C\right)=\operatorname{ctg} x-\frac{\operatorname{ctg}^3 x}{3}+x+C$$

$$\int \operatorname{ctg}^5 x\,dx=\left[\begin{array}{l} t=\operatorname{ctg} x\\ x=\operatorname{arcctg} t+\pi k,\ \ k\in Z\\ dx=-\dfrac{dt}{1+t^2}\end{array}\right]=-\int\frac{t^5 dt}{1+t^2}=-\int\left(t^3-t+\frac{t}{1+t^2}\right)dt=$$
$$=-\int t^3 dt+\int t\,dt-\frac{1}{2}\cdot\int\frac{d\left(1+t^2\right)}{1+t^2}=-\frac{t^4}{4}+\frac{t^2}{2}-\frac{1}{2}\ln\left|1+t^2\right|+C=$$
$$=-\frac{\operatorname{ctg}^4 x}{4}+\frac{\operatorname{ctg}^2 x}{2}-\frac{1}{2}\ln\left|1+\operatorname{ctg}^2 x\right|+C=\frac{\operatorname{ctg}^2 x}{2}\cdot\left(1-\frac{\operatorname{ctg}^2 x}{2}\right)-\frac{1}{2}\ln\left|\frac{1}{\sin^2 x}\right|+C=$$
$$=\frac{\operatorname{ctg}^2 x}{2}\cdot\left(1-\frac{\operatorname{ctg}^2 x}{2}\right)+\ln|\sin x|+C$$

$$\int \operatorname{ctg}^6 x\,dx=\left[\begin{array}{l} t=\operatorname{ctg} x\\ x=\operatorname{arcctg} t+\pi k,\ \ k\in Z\\ dx=-\dfrac{dt}{1+t^2}\end{array}\right]=-\int\frac{t^6 dt}{1+t^2}=-\int\left(t^4-t^2+1-\frac{1}{1+t^2}\right)dt=$$
$$=-\frac{t^5}{5}+\frac{t^3}{3}-t-\int d(\operatorname{arcctg} t)=-\frac{t^5}{5}+\frac{t^3}{3}-t-\operatorname{arcctg} t+C_1=$$
$$=-\frac{\operatorname{ctg}^5 x}{5}+\frac{\operatorname{ctg}^3 x}{3}-\operatorname{ctg} x-x+\pi k+C_1=-x-\frac{\operatorname{ctg}^5 x}{5}+\frac{\operatorname{ctg}^3 x}{3}-\operatorname{ctg} x+C$$

$$\int \frac{dx}{\operatorname{ctg}^6 x} = \int \frac{\sin^6 x\,dx}{\cos^6 x} = \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t \\ dx = \dfrac{dt}{1+t^2} \\ \sin^2 x = \dfrac{t^2}{1+t^2} \\ \cos^2 x = \dfrac{1}{1+t^2} \end{array} \right] = \int \frac{t^6 dt}{t^2+1} = \int \left(t^4 - t^2 + 1 - \frac{1}{t^2+1} \right) dt =$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + t - \operatorname{arctg} t + C = \frac{\operatorname{tg}^5 x}{5} - \frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x - t + C$$

Вычисление некоторых интегралов в Mathcad 14:

$$\int \tan(x)\,dx \rightarrow -\ln(\cos(x)) \qquad \int \tan(x)^4\,dx \rightarrow \frac{\tan(x)^3}{3} - \tan(x) + x$$

$$\int \cot(x)^2\,dx \rightarrow -x - \cot(x) \qquad \int \tan(x)^5\,dx \rightarrow \frac{\tan(x)^4}{4} - \frac{\tan(x)^2}{2} - \ln(\cos(x))$$

$$\int \tan(x)^3\,dx \rightarrow \frac{\tan(x)^2}{2} + \ln(\cos(x)) \qquad \int \cot(x)^6\,dx \rightarrow \frac{\cot(x)^3}{3} - \frac{\cot(x)^5}{5} - \cot(x) - x$$

Разложение на множители неправильной рациональной алгебраической дроби:

$$\frac{t^4}{1+t^2} \text{ parfrac } \rightarrow \frac{1}{t^2+1} + t^2 - 1 \qquad \frac{t^6}{1+t^2} \text{ parfrac } \rightarrow t^4 - t^2 - \frac{1}{t^2+1} + 1$$

$$\frac{t^5}{1+t^2} \text{ parfrac } \rightarrow \frac{t}{t^2+1} - t + t^3$$