

Примеры вычисления интегралов, содержащих тригонометрические функции $\operatorname{tg} x$ и $\operatorname{ctg} x$.

$$\begin{array}{cccccc} \int \operatorname{tg} x \, dx & \int \operatorname{tg}^2 x \, dx & \int \operatorname{tg}^3 x \, dx & \int \operatorname{tg}^4 x \, dx & \int \operatorname{tg}^5 x \, dx & \int \operatorname{tg}^6 x \, dx \\ \int \operatorname{ctg} x \, dx & \int \operatorname{ctg}^2 x \, dx & \int \operatorname{ctg}^3 x \, dx & \int \operatorname{ctg}^4 x \, dx & \int \operatorname{ctg}^5 x \, dx & \int \operatorname{ctg}^6 x \, dx & \int \frac{dx}{\operatorname{ctg}^6 x} \end{array}$$

$$\int \operatorname{tg} x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{d(\cos x)}{\cos x} = -\ln |\cos x| + C$$

$$\int \operatorname{tg}^2 x \, dx = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \int \frac{dx}{\cos^2 x} - \int dx = \operatorname{tg} x - x + C$$

$$\begin{aligned} \int \operatorname{tg}^3 x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} dx = \int \frac{\sin^2 x}{\cos^3 x} \cdot \sin x \, dx = -\int \frac{(1 - \cos^2 x)}{\cos^3 x} d(\cos x) = \int \left(\frac{1}{\cos x} - \frac{1}{\cos^3 x} \right) d(\cos x) = \\ &= \ln |\cos x| + \frac{1}{2 \cos^2 x} + C_1 = \ln |\cos x| + \frac{1}{2} \cdot (\operatorname{tg}^2 x + 1) + C_1 = \ln |\cos x| + \frac{\operatorname{tg}^2 x}{2} + C \end{aligned}$$

$$\begin{aligned} \int \operatorname{tg}^4 x \, dx &= \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t + \pi k, \quad k \in \mathbb{Z} \\ dx = \frac{dt}{1+t^2} \end{array} \right] = \int \frac{t^4 dt}{1+t^2} = \int \left(-1 + t^2 + \frac{1}{1+t^2} \right) dt = -t + \frac{t^3}{3} + \operatorname{arctg} t + C_1 = \\ &= -\operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + x - \pi k + C_1 = x - \operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + C \end{aligned}$$

$$\begin{aligned} \int \operatorname{tg}^5 x \, dx &= \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t + \pi k, \quad k \in \mathbb{Z} \\ dx = \frac{dt}{1+t^2} \end{array} \right] = \int \frac{t^5 dt}{1+t^2} = \int \left(t^3 - t + \frac{t}{1+t^2} \right) dt = \\ &= \int t^3 dt - \int t dt + \frac{1}{2} \cdot \int \frac{d(1+t^2)}{1+t^2} = \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2} \ln |1+t^2| + C = \frac{\operatorname{tg}^4 x}{4} - \frac{\operatorname{tg}^2 x}{2} + \frac{1}{2} \ln |1 + \operatorname{tg}^2 x| + C = \\ &= \frac{\operatorname{tg}^2 x}{2} \cdot \left(\frac{\operatorname{tg}^2 x}{2} - 1 \right) + \frac{1}{2} \ln \left| \frac{1}{\cos^2 x} \right| + C = \frac{\operatorname{tg}^2 x}{2} \cdot \left(\frac{\operatorname{tg}^2 x}{2} - 1 \right) - \ln |\cos x| + C \end{aligned}$$

$$\begin{aligned} \int \operatorname{tg}^6 x \, dx &= \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t + \pi k, \quad k \in \mathbb{Z} \\ dx = \frac{dt}{1+t^2} \end{array} \right] = \int \frac{t^6 dt}{1+t^2} = \int \left(t^4 - t^2 + 1 - \frac{1}{1+t^2} \right) dt = \\ &= \frac{t^5}{5} - \frac{t^3}{3} + t - \int d(\operatorname{arctg} t) = \frac{t^5}{5} - \frac{t^3}{3} + t - \operatorname{arctg} t + C_1 = \\ &= \frac{\operatorname{tg}^5 x}{5} - \frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x - x + \pi k + C_1 = -x + \frac{\operatorname{tg}^5 x}{5} - \frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x + C \end{aligned}$$

$$\int \operatorname{ctg} x \, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = \ln |\sin x| + C$$

$$\int \operatorname{ctg}^2 x \, dx = \int \left(\frac{1}{\sin^2 x} - 1 \right) dx = \int \frac{dx}{\sin^2 x} - \int dx = -\operatorname{ctg} x - x + C$$

$$\begin{aligned} \int \operatorname{ctg}^3 x \, dx &= \int \frac{\cos^3 x}{\sin^3 x} dx = \int \frac{\cos^2 x}{\sin^3 x} \cdot \cos x \, dx = \int \frac{(1 - \sin^2 x)}{\sin^3 x} d(\sin x) = \int \left(\frac{1}{\sin^3 x} - \frac{1}{\sin x} \right) d(\sin x) = \\ &= -\frac{1}{2\sin^2 x} - \ln |\sin x| + C_1 = -\frac{1}{2} \cdot (\operatorname{ctg}^2 x + 1) - \ln |\sin x| + C_1 = -\frac{\operatorname{ctg}^2 x}{2} - \ln |\sin x| + C \end{aligned}$$

$$\begin{aligned} \int \operatorname{ctg}^4 x \, dx &= \left[\begin{array}{l} t = \operatorname{ctg} x \\ x = \operatorname{arccotg} t + \pi k, \quad k \in \mathbb{Z} \\ dx = -\frac{dt}{1+t^2} \end{array} \right] = -\int \frac{t^4 dt}{1+t^2} = -\int \left(t^2 - 1 + \frac{1}{1+t^2} \right) dt = \\ &= -\left(\frac{t^3}{3} - t - \operatorname{arccotg} t + \pi k - C \right) = \operatorname{ctg} x - \frac{\operatorname{ctg}^3 x}{3} + x + C \end{aligned}$$

$$\begin{aligned} \int \operatorname{ctg}^5 x \, dx &= \left[\begin{array}{l} t = \operatorname{ctg} x \\ x = \operatorname{arccotg} t + \pi k, \quad k \in \mathbb{Z} \\ dx = -\frac{dt}{1+t^2} \end{array} \right] = -\int \frac{t^5 dt}{1+t^2} = -\int \left(t^3 - t + \frac{t}{1+t^2} \right) dt = \\ &= -\int t^3 dt + \int t dt - \frac{1}{2} \cdot \int \frac{d(1+t^2)}{1+t^2} = -\frac{t^4}{4} + \frac{t^2}{2} - \frac{1}{2} \ln |1+t^2| + C = \\ &= -\frac{\operatorname{ctg}^4 x}{4} + \frac{\operatorname{ctg}^2 x}{2} - \frac{1}{2} \ln |1 + \operatorname{ctg}^2 x| + C = \frac{\operatorname{ctg}^2 x}{2} \cdot \left(1 - \frac{\operatorname{ctg}^2 x}{2} \right) - \frac{1}{2} \ln \left| \frac{1}{\sin^2 x} \right| + C = \\ &= \frac{\operatorname{ctg}^2 x}{2} \cdot \left(1 - \frac{\operatorname{ctg}^2 x}{2} \right) + \ln |\sin x| + C \end{aligned}$$

$$\begin{aligned} \int \operatorname{ctg}^6 x \, dx &= \left[\begin{array}{l} t = \operatorname{ctg} x \\ x = \operatorname{arccotg} t + \pi k, \quad k \in \mathbb{Z} \\ dx = -\frac{dt}{1+t^2} \end{array} \right] = -\int \frac{t^6 dt}{1+t^2} = -\int \left(t^4 - t^2 + 1 - \frac{1}{1+t^2} \right) dt = \\ &= -\frac{t^5}{5} + \frac{t^3}{3} - t - \int d(\operatorname{arccotg} t) = -\frac{t^5}{5} + \frac{t^3}{3} - t - \operatorname{arccotg} t + C_1 = \\ &= -\frac{\operatorname{ctg}^5 x}{5} + \frac{\operatorname{ctg}^3 x}{3} - \operatorname{ctg} x - x + \pi k + C_1 = -x - \frac{\operatorname{ctg}^5 x}{5} + \frac{\operatorname{ctg}^3 x}{3} - \operatorname{ctg} x + C \end{aligned}$$

$$\int \frac{dx}{\operatorname{ctg}^6 x} = \int \frac{\sin^6 x dx}{\cos^6 x} = \left[\begin{array}{l} t = \operatorname{tg} x \\ x = \operatorname{arctg} t \\ dx = \frac{dt}{1+t^2} \\ \sin^2 x = \frac{t^2}{1+t^2} \\ \cos^2 x = \frac{1}{1+t^2} \end{array} \right] = \int \frac{t^6 dt}{t^2+1} = \int \left(t^4 - t^2 + 1 - \frac{1}{t^2+1} \right) dt =$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + t - \operatorname{arctg} t + C = \frac{\operatorname{tg}^5 x}{5} - \frac{\operatorname{tg}^3 x}{3} + \operatorname{tg} x - t + C$$

Вычисление некоторых интегралов в Mathcad 14:

$$\begin{array}{ll} \int \tan(x) dx \rightarrow -\ln(\cos(x)) & \int \tan(x)^4 dx \rightarrow \frac{\tan(x)^3}{3} - \tan(x) + x \\ \int \cot(x)^2 dx \rightarrow -x - \cot(x) & \int \tan(x)^5 dx \rightarrow \frac{\tan(x)^4}{4} - \frac{\tan(x)^2}{2} - \ln(\cos(x)) \\ \int \tan(x)^3 dx \rightarrow \frac{\tan(x)^2}{2} + \ln(\cos(x)) & \int \cot(x)^6 dx \rightarrow \frac{\cot(x)^3}{3} - \frac{\cot(x)^5}{5} - \cot(x) - x \end{array}$$

Разложение на множители неправильной рациональной алгебраической дроби:

$$\begin{array}{ll} \frac{t^4}{1+t^2} \operatorname{parfrac} \rightarrow \frac{1}{t^2+1} + t^2 - 1 & \frac{t^6}{1+t^2} \operatorname{parfrac} \rightarrow t^4 - t^2 - \frac{1}{t^2+1} + 1 \\ \frac{t^5}{1+t^2} \operatorname{parfrac} \rightarrow \frac{t}{t^2+1} - t + t^3 & \end{array}$$